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A METHOD OF SOFT TETHER STATIONKEEPING

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ABSTRACT

This report considers the motion of one spacecraft in a near-circular orbit which is fastened to another with a flexible tether line. The equations of motion are obtained through a closed form solution to the approximate differential equation. A method is devised by which pulls can be made on the tether in a repeatable sequence so that the two spacecraft will remain in the vicinity of one another when there are differential drag forces acting on them. This method is then applied to the case of a Lunar Module/Apollo Telescope Mount (LM/ATM) tethered to an S-IVB Orbital Workshop (OWS).

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MISSION ANALYSIS BRANCH
DYNAMICS AND FLIGHT MECHANICS DIVISION
AERO-ASTRODYNAMICS LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

LIST OF ILLUSTRATIONS

TECHNICAL MEMORANDUM X-53643

A METHOD OF SOFT TETHER STATIONKEEPING

SUMMARY

This report considers the motion of one spacecraft in a near-circular orbit which is fastened to another with a flexible tether line. The equations of motion are obtained through a closed form solution to the approximate differential equation. A method is devised by which pulls can be made on the tether in a repeatable sequence so that the two spacecraft will remain in the vicinity of one another when there are differential drag forces acting on them. This method is then applied to the case of a Lunar Module/Apollo Telescope Mount (LM/ATM) tethered to an S-IVB Orbital Workshop (OWS).

I. INTRODUCTION

With the advent of more sophisticated space missions, the need to maintain multiple spacecraft in close proximity to one another arises. Intuitively, it seems that there would be no problem caused by the extremely small differential forces acting on these spacecraft arising from small variations in venting thrusts, solar pressure and drag decelerations. However, as figure 1 shows, these forces, typical of those acting on two spacecraft of differing configuration and orientation, can cause large separation distances after several days' time. For this particular figure, the difference in the drag deceleration between the two bodies was $.312 \times 10^{-6}$ m/sec^2.

There is a variety of methods which may be employed to prevent the separation of spacecraft in orbit. They may be mechanically fixed together (hard docked) or they may be unattached and kept together by propulsive means. A compromise between these methods is to fix the bodies together with a tether line. Each of these methods has advantages and disadvantages depending on the goals of the particular mission. For missions which are propellant-limited but require that two initially closely spaced vehicles have independent attitudes, the tether method looks attractive enough for a more detailed analysis.

The tether method can be divided into two classes. One of these is characterized by a continuous tension in the tether. In the other class, the tether is taut at only discrete time points and is slack the rest of the time. This report analyzes the relative motion of two spacecraft for the latter class and presents a method for controlling their relative positions within desired bounds. No attempt is made to describe the methods which may be used for deployment of retrieval [1] of such tethered satellites.

To make an analysis of the spacecraft motion, first, the differential equation of motion of one spacecraft relative to the other is linearized and solved in closed form. This solution is then used to find a method by which the two spacecraft can be kept together. Finally, the physical characteristics of this method are described for the example of a Lunar Module/Apollo Telescope Mount tethered to an S-IVB Orbital Workshop.

II. DERIVATION OF THE EQUATIONS OF MOTION

Let one of the spacecraft be in a circular orbit of radius, R, and angular velocity, $\bar{\omega}$, (figure 2). Set up a two-dimensional coordinate system in this spacecraft with the positive y-axis pointing outward along the radius vector and the x-axis in the opposite direction of the inertial velocity vector. Let the position vector in this coordinate system to a second spacecraft be $\bar{r}$. If $\bar{R}$ is the radius vector from the center of the earth to the origin of the relative rotating coordinate system, then the differential equation of motion for the center of mass of the second body can be written:

$$\ddot{\bar{r}} = \bar{a} - 2\bar{\omega} \times \dot{\bar{r}} + \bar{g} - \bar{\omega} \times [\bar{\omega} \times (\bar{r} + \bar{R})],$$

where $\bar{a}$ is the sum of all the differential accelerations between the two bodies due to aerodynamic, propulsive and tether forces, $-2\bar{\omega} \times \dot{\bar{r}}$ is the Coriolis acceleration, $\bar{g}$ is the gravitational acceleration, and the last term is the centrifugal acceleration. The last two terms can be combined in the following manner:

$$\bar{g} = - \frac{\mu(\bar{r} + \bar{R})}{(r + R)^3},$$

according to the inverse square gravitational law, where μ is the gravitational constant for the earth. Since $\omega^2 = \mu/R^3$,

$$\bar{g} = \frac{-\omega^2(\bar{r} + \bar{R})\ R^3}{(r + R)^3} .$$

Since $\bar{\omega}$ is perpendicular to $(\bar{r} + \bar{R})$,

$$\bar{\omega}\times\ [\bar{\omega}\times(\bar{r}\times\bar{R})] = -\omega^2(\bar{r} + \bar{R})$$

and

$$\bar{g} - \bar{\omega}\times[\bar{\omega} + (\bar{r} + \bar{R})] = \omega^2(\bar{r} + \bar{R}) \left(1 - \frac{R^3}{(r + R)^3}\right).$$

The term $(r + R)$ can be approximated by $R + y$ with an error of only about x^2/R, where x and y are the components of $\bar{r}$ in figure 2. Therefore,

$$\bar{g} - \bar{\omega}\times[\bar{\omega}\times(\bar{r} + \bar{R})] = \omega^2(\bar{r} + \bar{R}) \left[1 - \frac{R^3}{(R + y)^3}\right]$$

$$= \omega^2(\bar{r} + \bar{R}) \left[1 - \frac{1}{(1 + \frac{y}{R})^3}\right].$$

Expanding the second term in the brackets by the binomial expansion yields

$$\bar{g} - \bar{\omega}\times[\bar{\omega}\times(\bar{r} + \bar{R})] = \omega^2(\bar{r} + \bar{R}) \left[1 - \left(1 - 3\,\frac{y}{R} + \ldots\right)\right]$$

$$\approx 3\omega^2 y\, \frac{(\bar{r} + \bar{R})}{R},$$

where $\frac{\bar{r} + \bar{R}}{R}$ may be approximated by the unit vector in the y direction, $\hat{j}$.

$$\ddot{\bar{r}} = \bar{a} - 2\omega\times\dot{\bar{r}} + 3\omega^2 y\hat{j}.$$

Separating this acceleration into x and y components yields

$$\ddot{x} = a_x + 2\omega\dot{y}$$

$$\ddot{y} = a_y - 2\omega\dot{x} + 3\omega^2 y.$$

For this report, it will be assumed that the difference in drag deceleration acts on the second body in the positive x-direction and is a constant, D. The Rand Corporation investigated the equations for a drag which is dependent on altitude [2]. However, this assumption is justified for very nearly circular orbits in a spherically symmetric atmosphere (with no diurnal bulge). During the time when there is no tension in the tether, these differential equations therefore become

$$\ddot{x} = D + 2\omega\dot{y} \tag{1}$$

$$\ddot{y} = -2\omega\dot{x} + 3\omega^2 y. \tag{2}$$

These equations may be solved analytically as shown in Appendix A to give x, y, $\dot{x}$, and $\dot{y}$ as functions of time.

$$x = x_o + c_o t - \frac{3}{2} Dt^2 - 2b_o \sin \omega t + 2a_o(1 - \cos \omega t) \tag{3}$$

$$y = \frac{2c_o}{3\omega} - \frac{2D}{\omega} t - b_o \cos \omega t + a_o \sin \omega t \tag{4}$$

$$\dot{x} = c_o - 3Dt - 2\omega b_o \cos \omega t + 2\omega a_o \sin \omega t \tag{5}$$

$$\dot{y} = - \frac{2D}{\omega} + \omega b_o \sin \omega t + \omega a_o \cos \omega t, \tag{6}$$

where

$$a_o = \frac{1}{\omega}\left(\dot{y}_o + \frac{2d}{\omega}\right),$$

$$b_o = 3y_o - 2\frac{\dot{x}_o}{\omega},$$

$$c_o = 6\omega y_o - 3\dot{x}_o,$$

and x_o, y_o, $\dot{x}_o$, and $\dot{y}_o$ are the conditions of position and velocity at time t = 0.

These equations are not restricted to use with tethered satellites. They may be used to describe the motion of any body in a near-circular orbit which remains fairly close (i.e., much less than the radius of the orbit) to the origin of the rotating relative coordinate system. For example, they could be used for the coasts between burns of the terminal maneuvers of rendezvous and docking. Also, these equations are not restricted to describing the motion of only one body. By having a separate set of equations as (3) through (6) for each body, the motion of multiple spacecraft in low eccentricity orbits could be described. By subtracting their coordinates, the motion of all of the bodies could be described relative to one of them. This is a possible method of removing the restriction that one of the spacecraft be exactly in a circular orbit.

III. SOLUTION TO THE STATIONKEEPING PROBLEM

In the following discussion, the body located at the origin of the relative coordinate system will be called the reference body and the other spacecraft the tethered body.

The equations of motion developed in the previous section cause the tethered spacecraft to follow a trajectory relative to the reference body similar to that shown in figure 3. In this particular example, the motion of the tethered body is initiated at the position $x = 30$ m, $y = 4$ m, with a positive $\dot{x}$ and $\dot{y}$ such that its orbital energy is greater than that of the reference body. Superimposed onto an oscillation in the vertical and horizontal direction, the tethered body first experiences a drift to the right and then a drift to the left. The oscillation in the vertical direction results as it moves between apogee and perigee. The oscillation in the horizontal direction is due to the fact that the perigee and apogee velocities are different. The drift to the right is caused by the semimajor axis of the conic of the tethered body being initially greater than that of the reference body and thus having a longer period. As drag decays the orbit, the semimajor axis decreases and the spacecraft then drifts to the left. The orbital energy lost through atmospheric drag must in some manner be regained, or the orbital period will continue to become less and less, thus increasing the separation rate. If energy is to be added to the tethered body by adding to the velocity by pulling on the tether, this must be done before it gets ahead of the reference body (since any pulls after it gets ahead will only decrease its velocity). It would also be highly desirable to be able to get on a transfer ellipse (such as indicated in figure 3 by the dashed line), and then when the spacecraft intersects the original trajectory, give a second impulse to reproduce the initial conditions. If this could be done, the process could be repeated over and over. Obviously, the correct impulses to do this could be found if the magnitude and direction of the pull could be controlled. However, the direction is restricted to lie in the direction of the taut tether. Since, in general, this will not be the desired direction, some constraint must be placed on the initial trajectory which will insure that the direction of the tether be correct to initiate the transfer and obtain a repeatable sequence. In Appendix B, it is found that a sufficient condition to insure repeatability is for the motion of the tethered body to describe a trajectory which is symmetric with respect to the x-axis in the relative coordinate system. Figure 4 shows a typical symmetric trajectory. Each one of those showing symmetry can be characterized by two arbitrary parameters. Let these be the initial altitude difference, y_{1i}, and the time, Δt_1, spent from the initial point until a pull on the tether is required. For any combination of these two, there can be found an initial velocity which will cause the motion to be symmetric. As shown in equations (B21) and (B22) in Appendix B, the components of this velocity are independent of the initial tangential separation, x_{1i}. However, there is a restriction on x_{1i} to be positive (the tethered body behind the reference body). This restriction is made to make equation (B31) solvable with a positive ΔV_1 since a negative ΔV_1 would be equivalent to a push with the tether.

In summary, the stationkeeping problem can be solved with a repeatable sequence by placing the tethered body on a symmetric trajectory.

IV. A PRACTICAL APPLICATION

To obtain greater insight into the characteristics of the stationkeeping problem and the use of symmetric trajectories as a solution, first a specific tether mission is considered. Then several of the variables pertinent to this mission are parameterized. Finally, the most attractive symmetric trajectory is chosen and described in more detail.

An S-IVB orbital workshop (OWS) [3] space station is in a 260-nautical-mile circular orbit. It is desired to keep a Lunar Module/ Apollo Telescope Mount (LM/ATM) [4] in the vicinity of the workshop by use of a tether line. As shown by the aerodynamic data in table I, the ATM will experience the greater drag deceleration due to orientation and ratio of drag to mass. The difference in drag acceleration, D, is about $.312 \times 10^{-6}$ m/sec^2.

The variables which were considered as being important or characteristic of the trajectories are the initial velocity components, the transfer time, Δt_2, the total cycle time, $\Delta t_1 + \Delta t_2$, and the amplitude of the perigee-apogee oscillation. The initial velocity components necessary to start the ATM on a symmetric trajectory can be obtained from equations (B21) and (B22) in Appendix B. Since these components are functions only of the altitude difference at the beginning of the symmetric trajectory, y_{1i}, and Δt_1, they can be represented as contours on an abstract plane, y_{1i}, Δt_1 (see figures 5 and 6). Similarly, the transfer time, Δt_2, characteristic for each symmetric trajectory may be plotted on this plane for some specified value of x_{1i}, the initial tangential separation. In figures 7 through 10, this has been done where $x_{1i} \gg y_{1i}$. Multiple transfer times are available for any symmetric trajectory due to the trigonometric part of equation (B33). By making the same assumption on x_{1i}, figures 11 through 14 show contours of equal total cycle time, $\Delta t_1 + \Delta t_2$. In this case, by multiplying by 1/2 D, these become contours of equal ΔV_1. For a cycle time of 4T, ΔV_1 is .003527 m/sec. To impart this velocity change to the LM/ATM, it is necessary to pull on the tether with a force of 1 pound for 5.86 seconds. This impulse being typical, only a very light tether line is required -- of the order of fishing line. The amplitude of the oscillation as the ATM moves between apogee and perigee may be derived as a function of y_{1i} and Δt_1 by using equation (B18). Contours of equal amplitude, a_o will obey the equation

$$y_{1i} = \pm a_o \sin \frac{1}{2} \omega \Delta t_1 + \frac{D}{\omega} \Delta t_1.$$

These contours are shown in figure 15. On an actual mission, Δt_1's near an integer number of orbital periods should be avoided because of large oscillations and because of velocities in the vertical direction.

The family of solutions indicated in figure 15 as having zero amplitude in the vertical direction is particularly interesting. The solid line in figure 16 shows the appearance of one of these case. In these cases, both the workshop and ATM are in circular orbits,* the ATM orbit initially being slightly higher. Since the period of the ATM is thus longer, it will drift further behind the workshop. Drag gradually lowers the altitude of the ATM orbit, decreasing its period. After its altitude drops below that of the workshop, it begins to catch up. When it has come to within a certain distance, x_{1f}, a tug is given which places it on a transfer trajectory to the initial conditions where a second pull recircularizes it. If $x_{1i} \gg (y_{1i})$, then the transfer is very close to a Hohmann transfer making Δt_2 about equal to half an orbital period.

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*The appearance of the trajectory would be very similar if the two bodies were in elliptical orbits of equal eccentricity and coincident lines of apsides.

The Δt_1 and the maximum distance between the ATM and workshop, x_{max}, which are obtained, and the $\dot{x}_{1i}$, which is necessary, are plotted as functions of y_{1i} in figure 17. The necessary $\dot{y}_{1i}$ is always equal to $-2D/\omega$. These cases are fairly insensitive to errors in both the timing and magnitude of the pulls on the tether. First, consider the pull which is made below the altitude of the workshop. If this pull is made somewhat early or late, the transfer back to the initial conditions will be performed slightly farther from or nearer to the workshop than the nominal. If the magnitude of the pull is not quite nominal, then the transfer will be to a somewhat different altitude. When the recircularization pull is made, the ATM will merely be on a different member of the family of zero oscillation cases indicated by the dashed lines in figure 16. Errors in the magnitude of the second pull as much as ± 10 percent and in timing as much as from 3 to 5 minutes introduce only a small amount of oscillatory motion, as indicated in figures 18 through 21.

V. CONCLUSIONS AND EXPANSIONS

It is possible to conclude from the ground work set up in this report that it is feasible to overcome the separation buildup between two satellites under the influence of differential drag forces by using a flexible tether line. This can be done by bringing the tether taut, imparting an impulse through it, and allowing the bodies to drift free before a second impulse is made. The sequence for pulling on the tether becomes repeatable if the motion of the tethered body appears symmetric relative to the reference body. The most practical of these symmetric cases seems to be a case which exhibited no oscillations.

Several areas which need further work are being investigated. The reduction of the equations in Appendix B should be generalized to establish the necessary conditions for a repeatable sequence.* Also, motion only in two dimensions has thus far been considered; this should be extended to motion out of the orbital plane. Errors in the timing and magnitude of the impulses should be further investigated, and some control laws should be developed to maintain a nominal sequence. Before making any actual flights, particularly when astronauts are a part of the mission, the safety hazards involved must be thoroughly studied. Not only must precautions to prevent hazardous situations be taken, but also remedies must be considered if these situations occur in spite of

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*The condition that the trajectories be symmetric is only a sufficient condition to insure repeatability.

the precautions. Certain unmanned experiments should be tried before a man is placed in the tethered body. Some effort is now being made by the Martin Company to determine the feasibility of making an actual flight experiment to test these soft tether techniques.

TABLE I

Drag Acceleration Data

	Workshop	ATM
Coefficient of Drag	10.15	3.77
Reference Area	33.478 m^2	33.478 m^2
Mass	28495 kg	7394 kg
m/C_DA	83.86 kg/m^2	58.58 kg/m^2

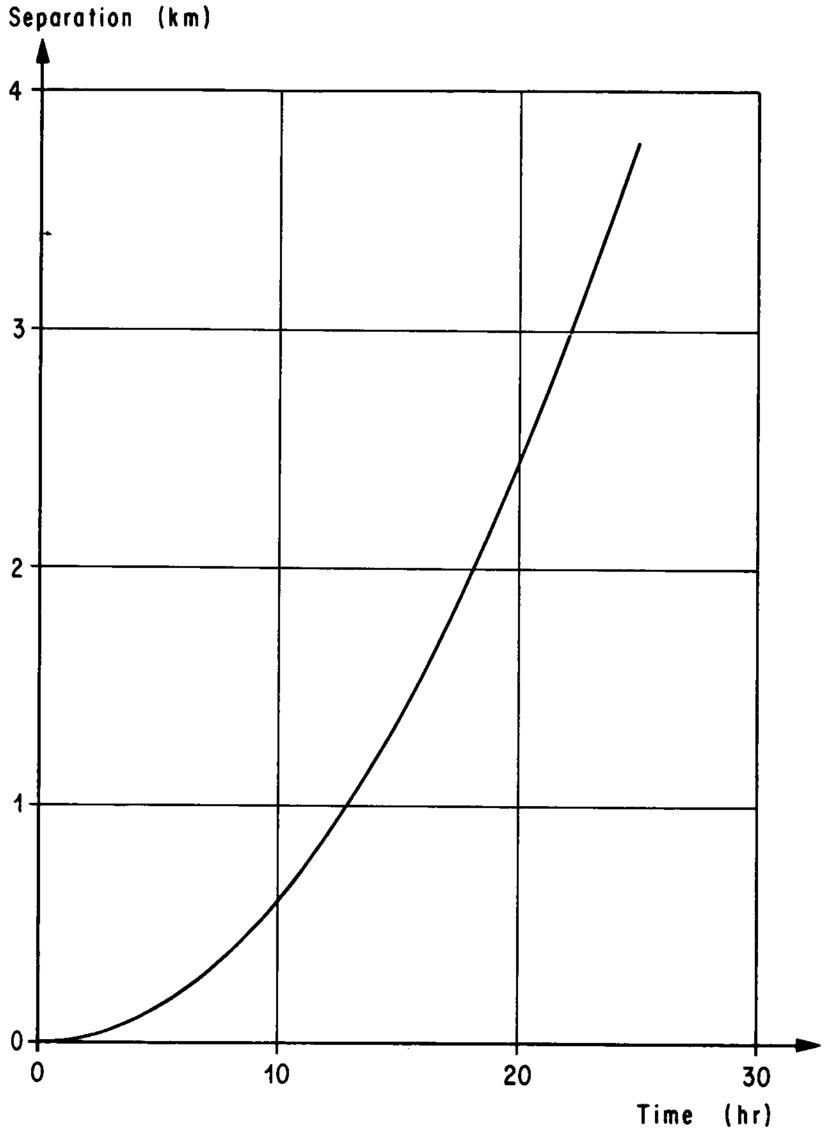

FIG. 1. TYPICAL SEPARATION GROWTH BETWEEN TWO SATELLITES IN LOW EARTH ORBIT

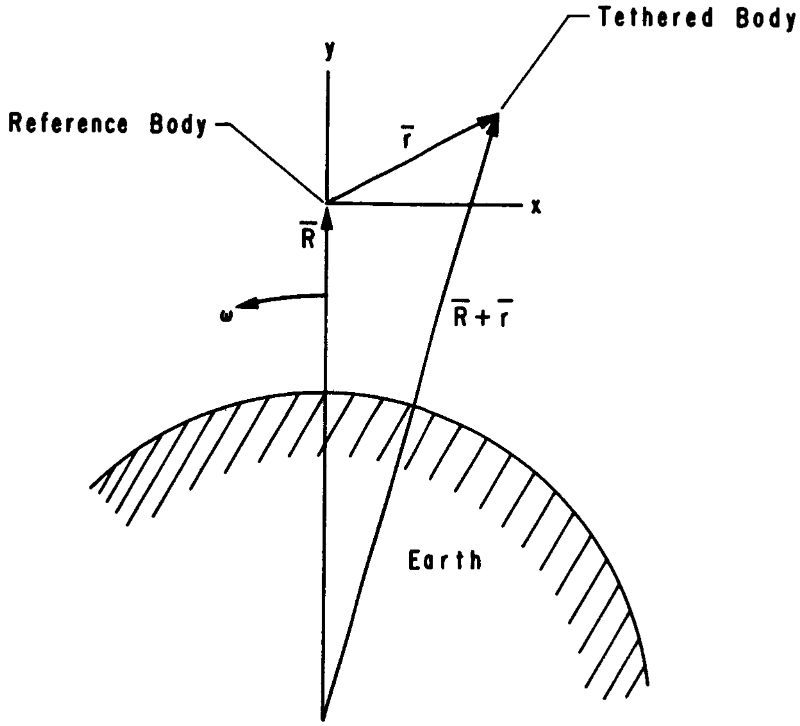

FIG. 2. RELATIVE COORDINATE SYSTEM

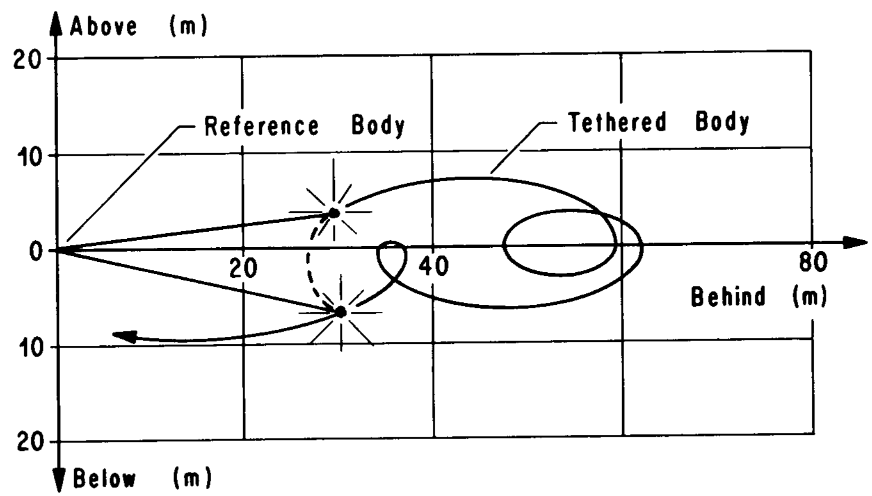

FIG. 3. TYPICAL TRAJECTORY IN THE RELATIVE COORDINATE SYSTEM

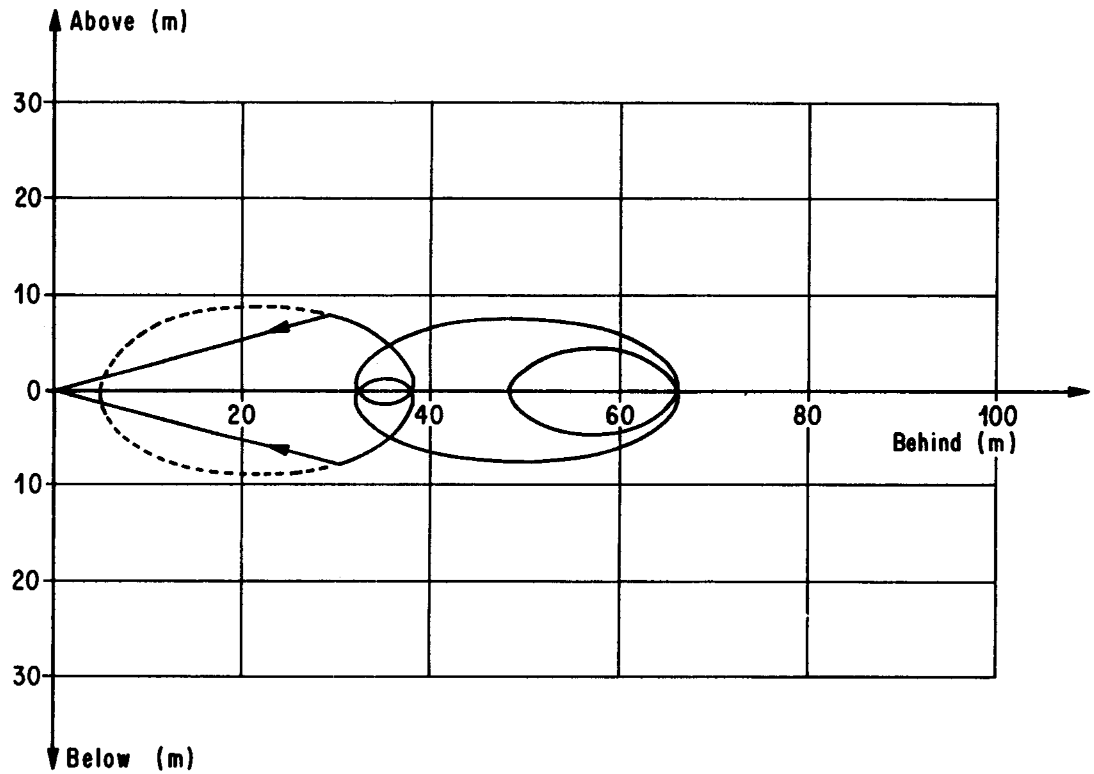

FIG. 4. A TYPICAL SYMMETRIC TRAJECTORY

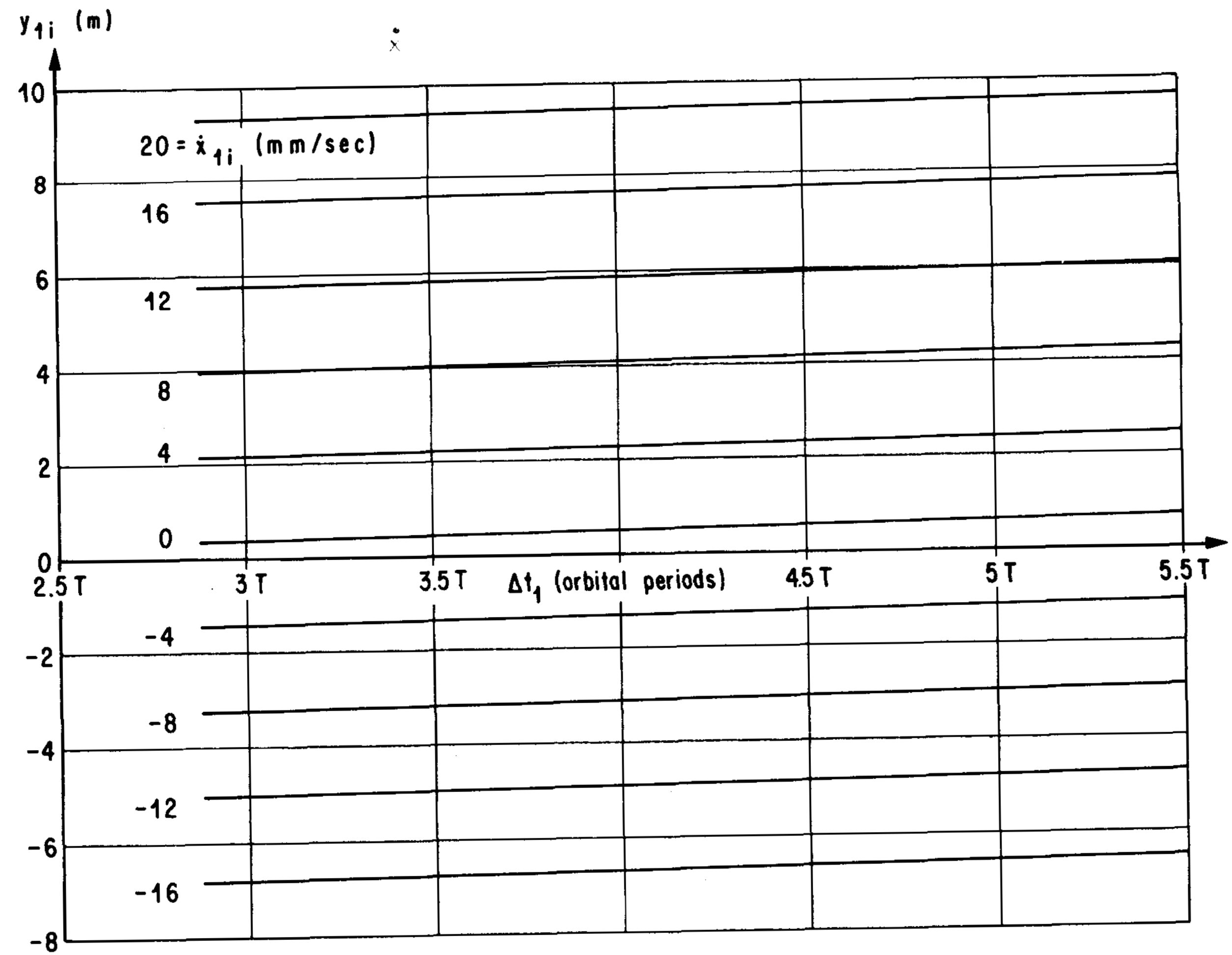

FIG. 5. CONTOURS OF EQUAL $\dot{x}_{1i}$ ON THE PLANE $(\Delta t_1, y_{1i})$

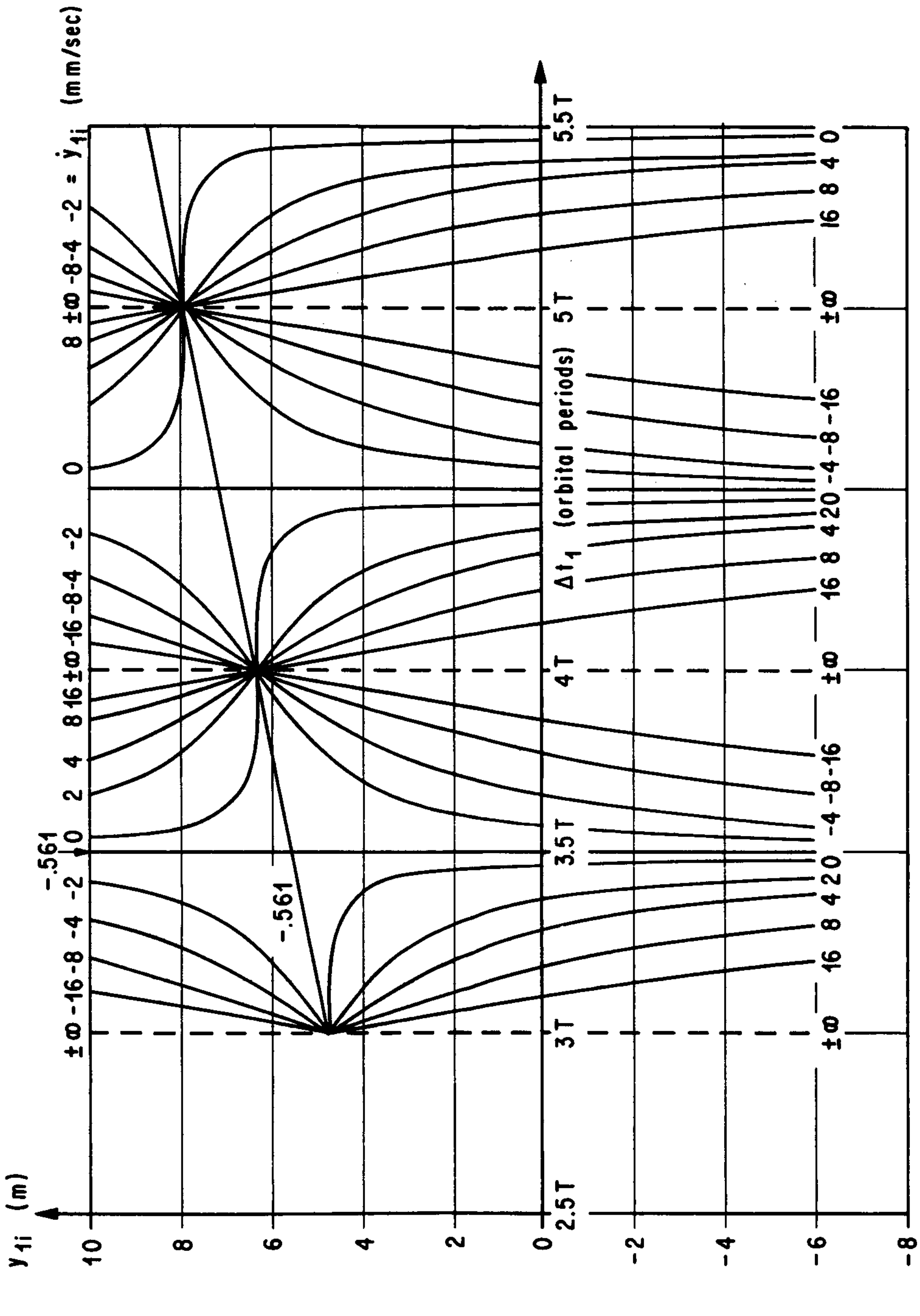

FIG. 6. CONTOURS OF EQUAL $\dot{y}_{1i}$ ON THE PLANE $(\Delta t_1, y_{1i})$

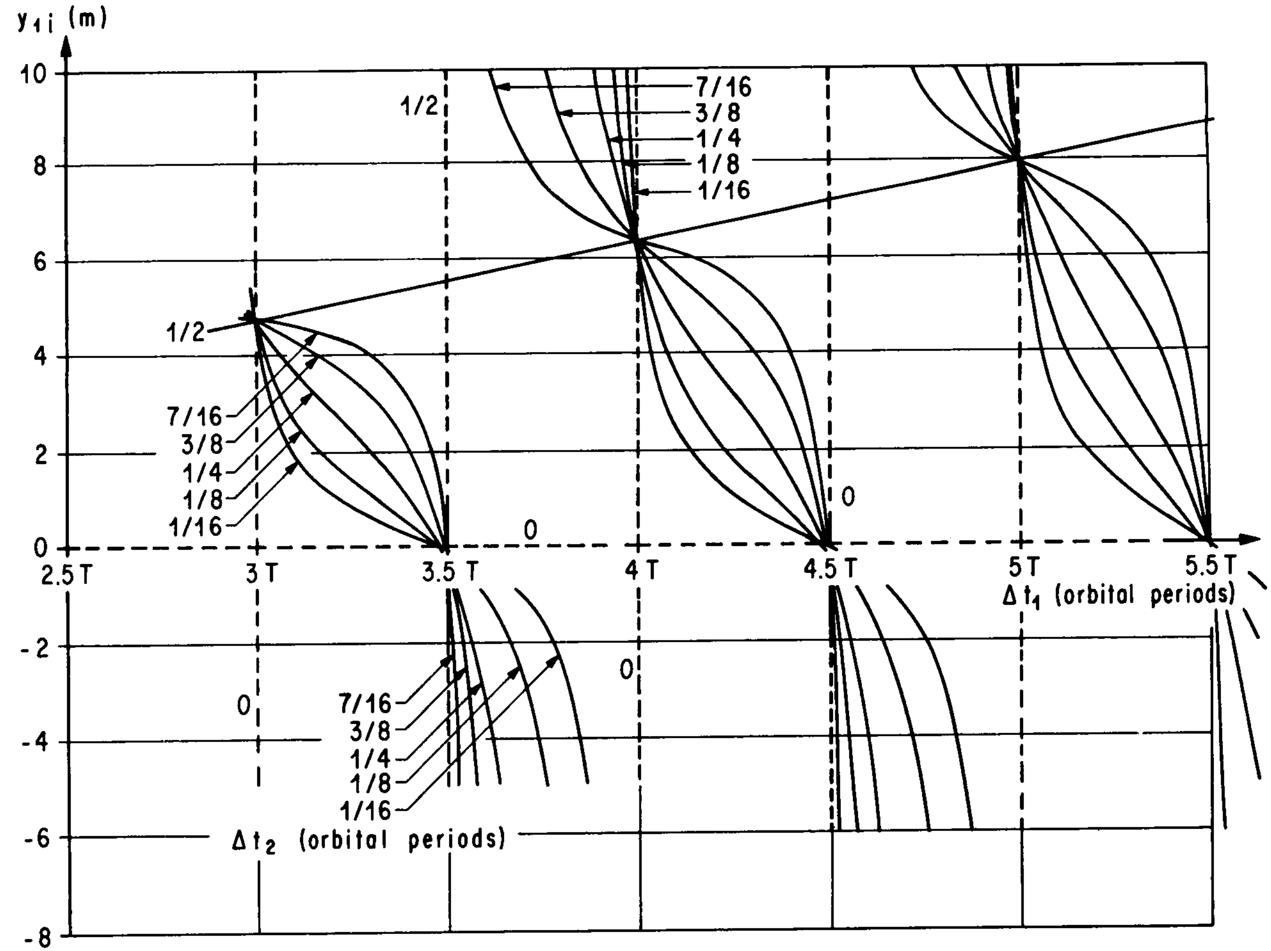

FIG. 7. CONTOURS OF EQUAL Δt_2 ON THE PLANE $(\Delta t_1, y_{1i})$

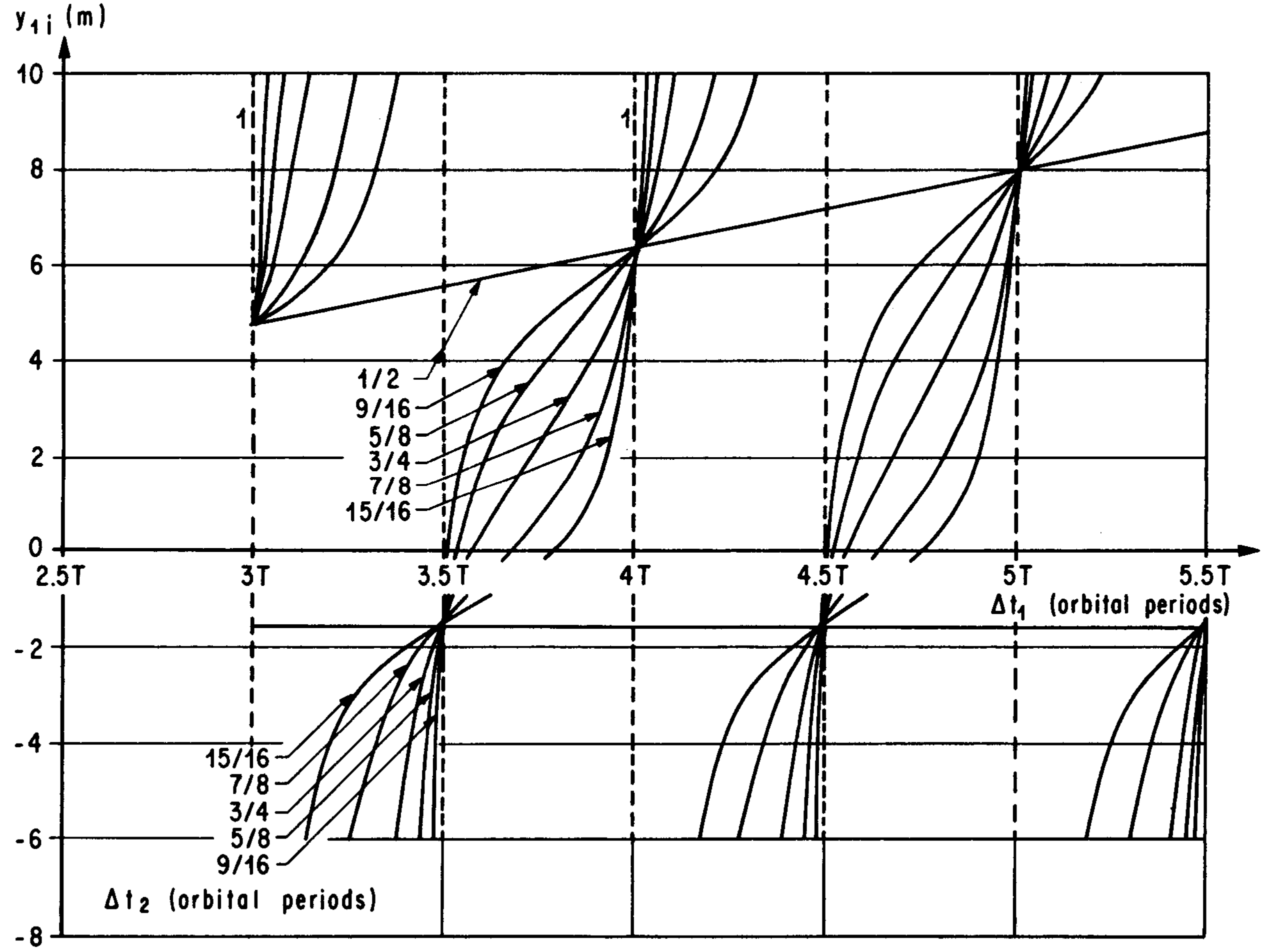

FIG. 8. CONTOURS OF EQUAL Δt_2 ON THE PLANE $(\Delta t_1, y_{1i})$

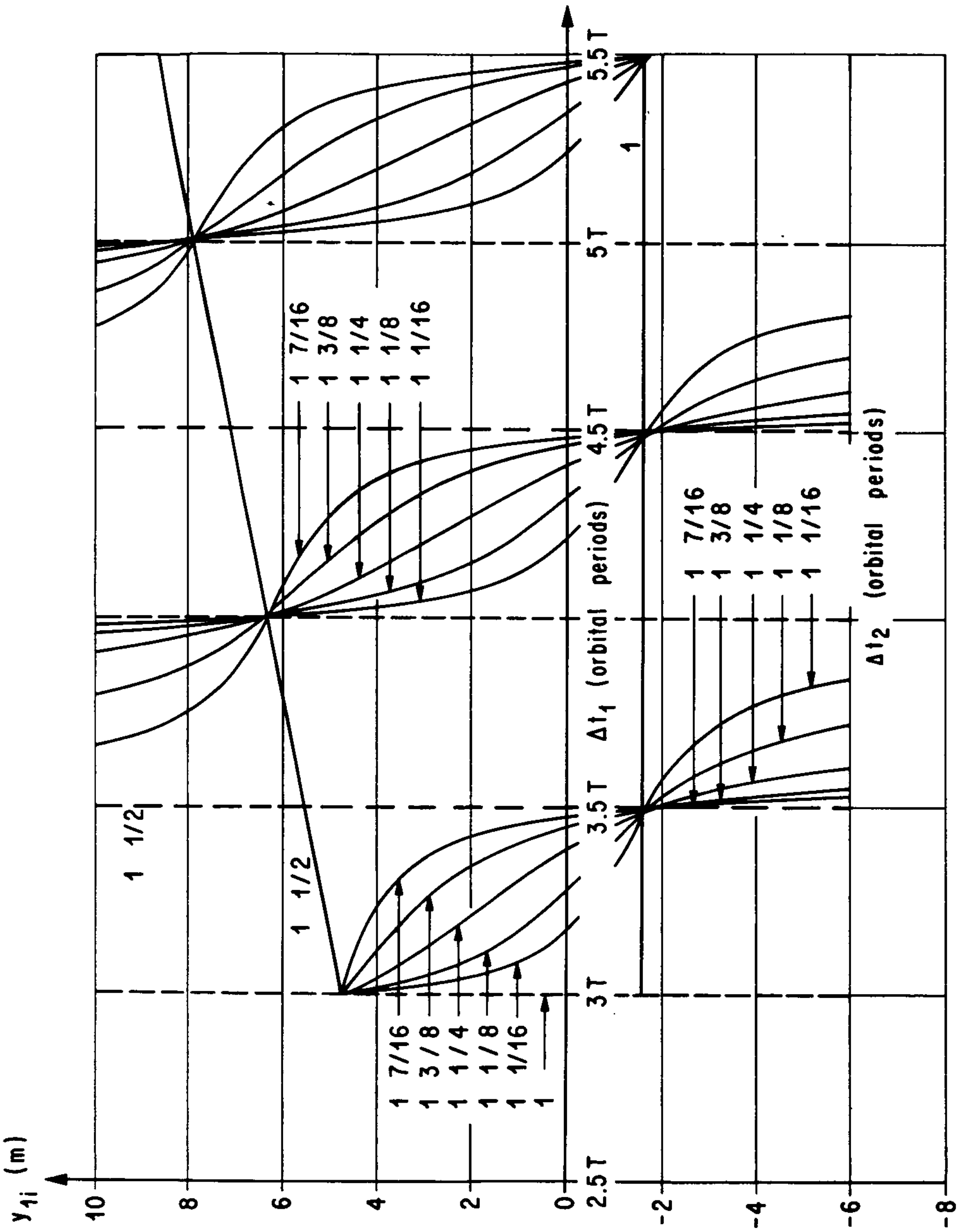

FIG. 9. CONTOURS OF EQUAL Δt_2 ON THE PLANE (Δt_1, y_{4i})

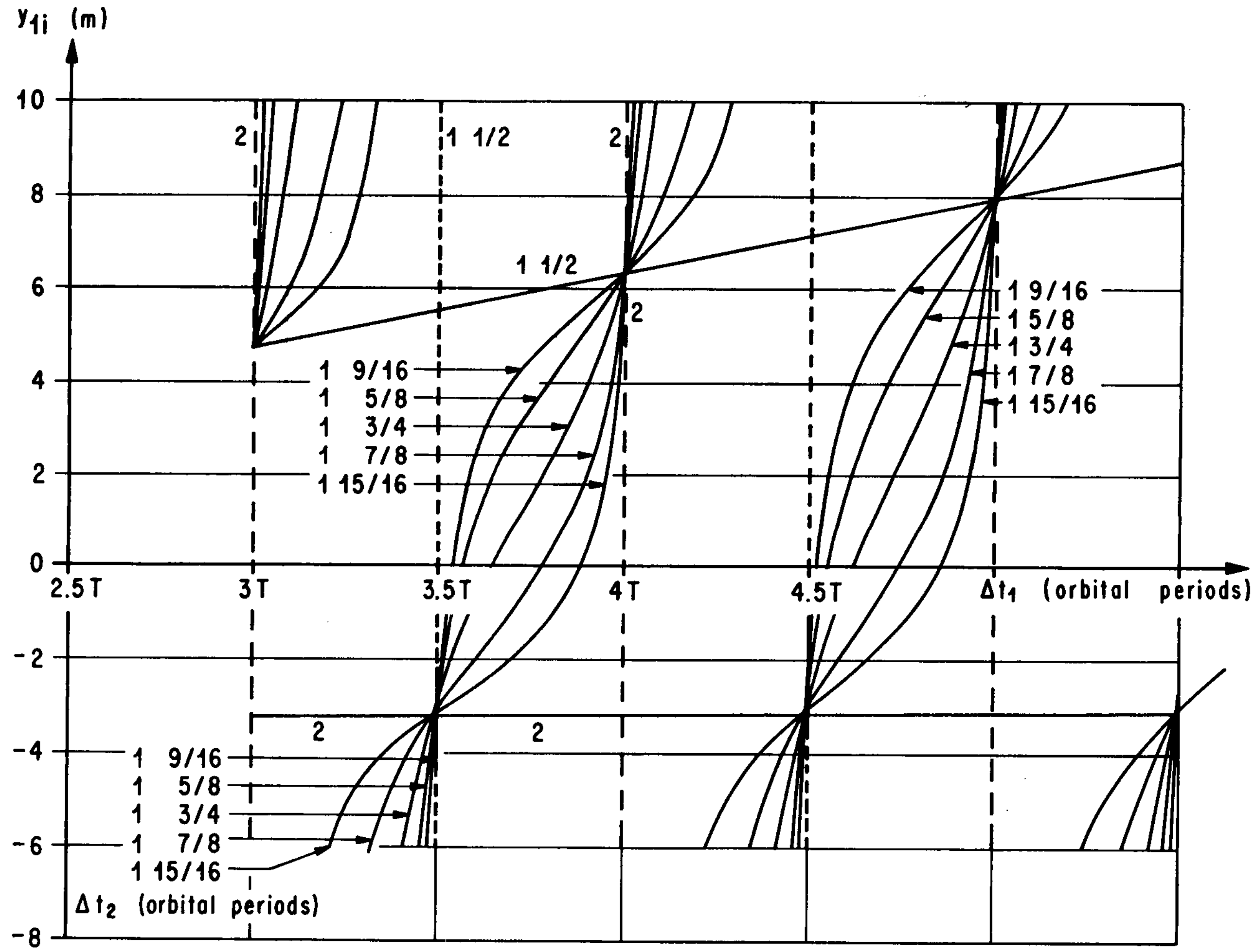

FIG. 10. CONTOURS OF EQUAL Δt_2 ON THE PLANE (Δt_1, y_{1i})

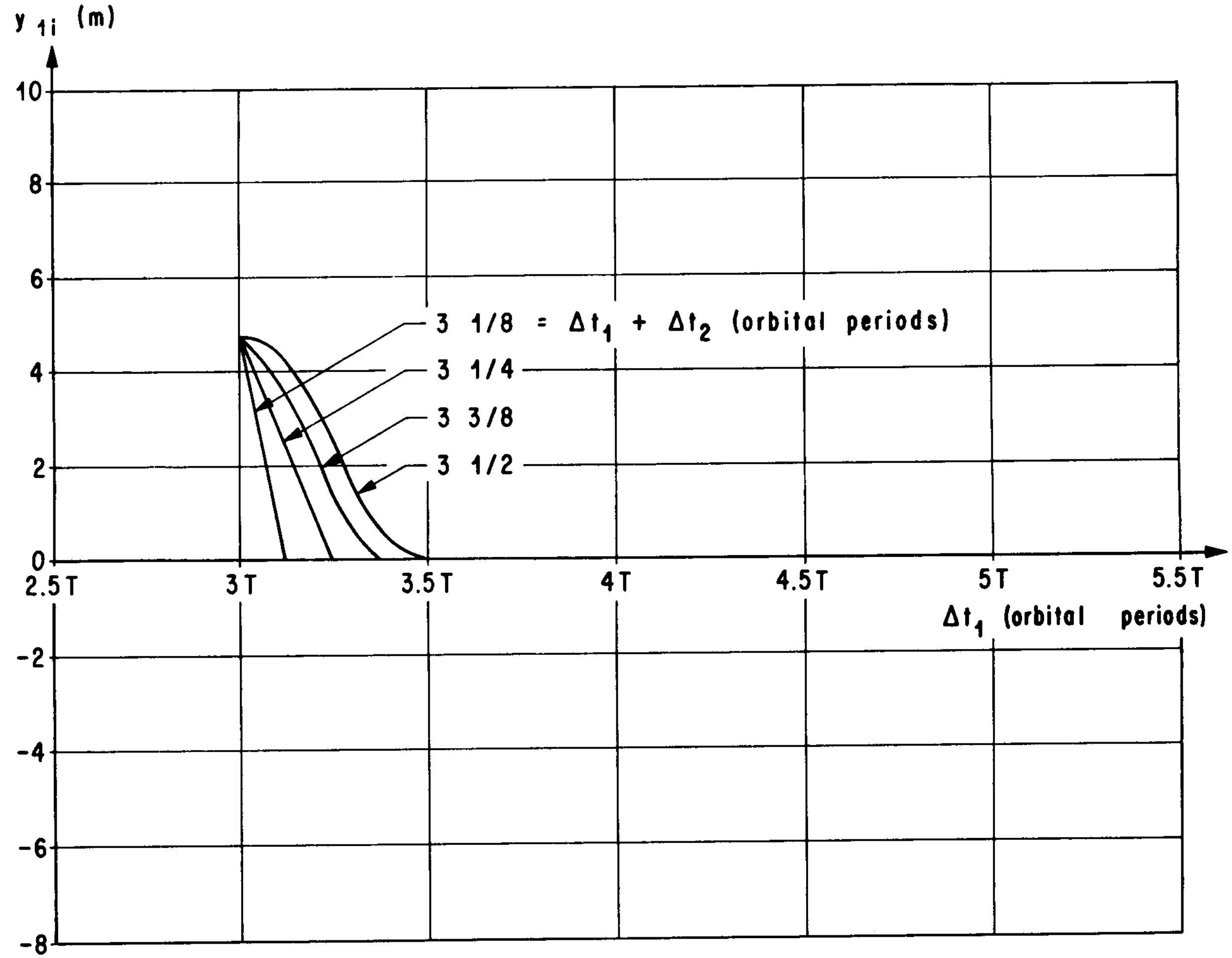

FIG. 11. CONTOURS OF EQUAL TOTAL CYCLE TIME ON THE PLANE $(\Delta t_1, y_{1i})$

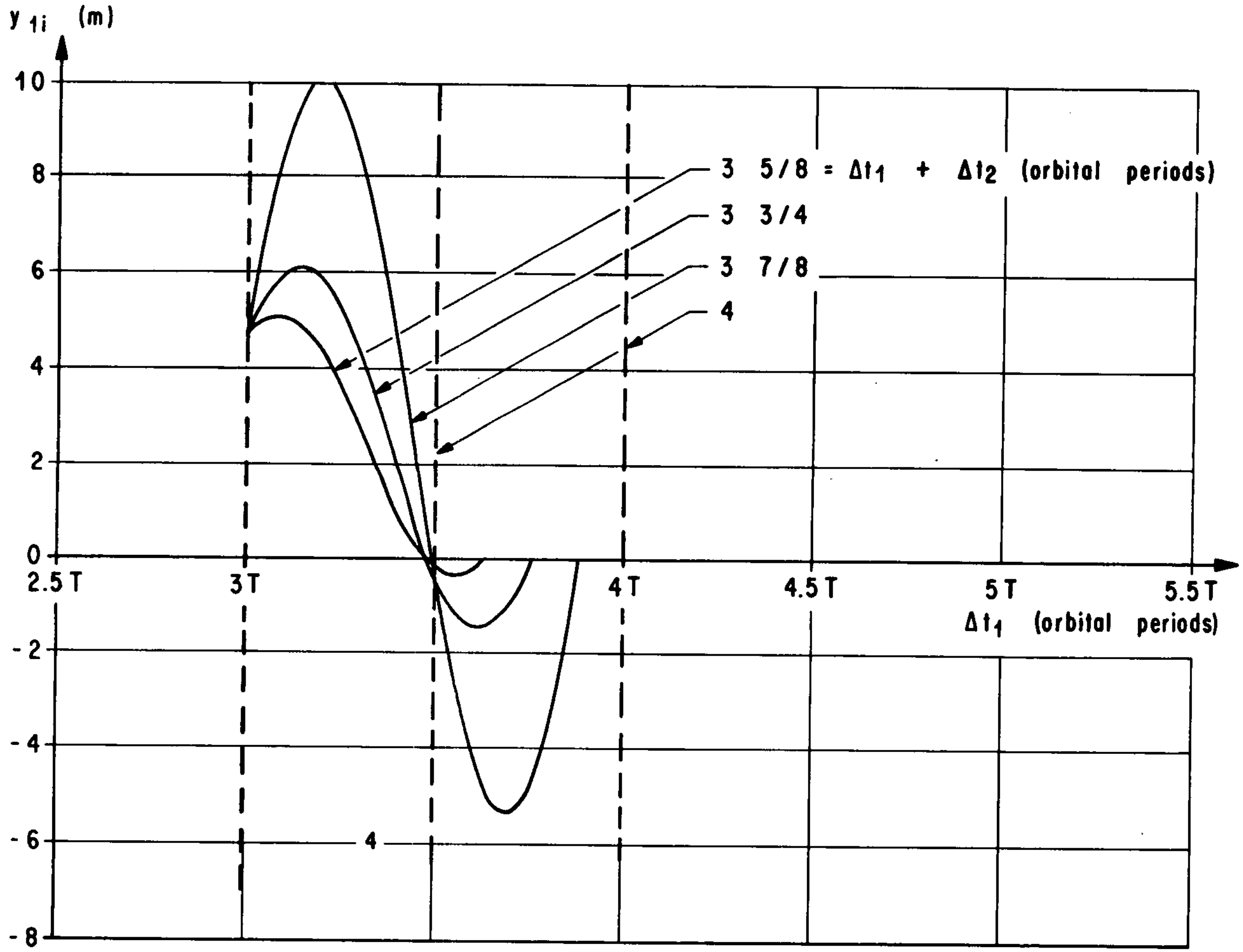

FIG. 12. CONTOURS OF EQUAL TOTAL CYCLE TIME ON THE PLANE $(\Delta t_1, y_{1i})$

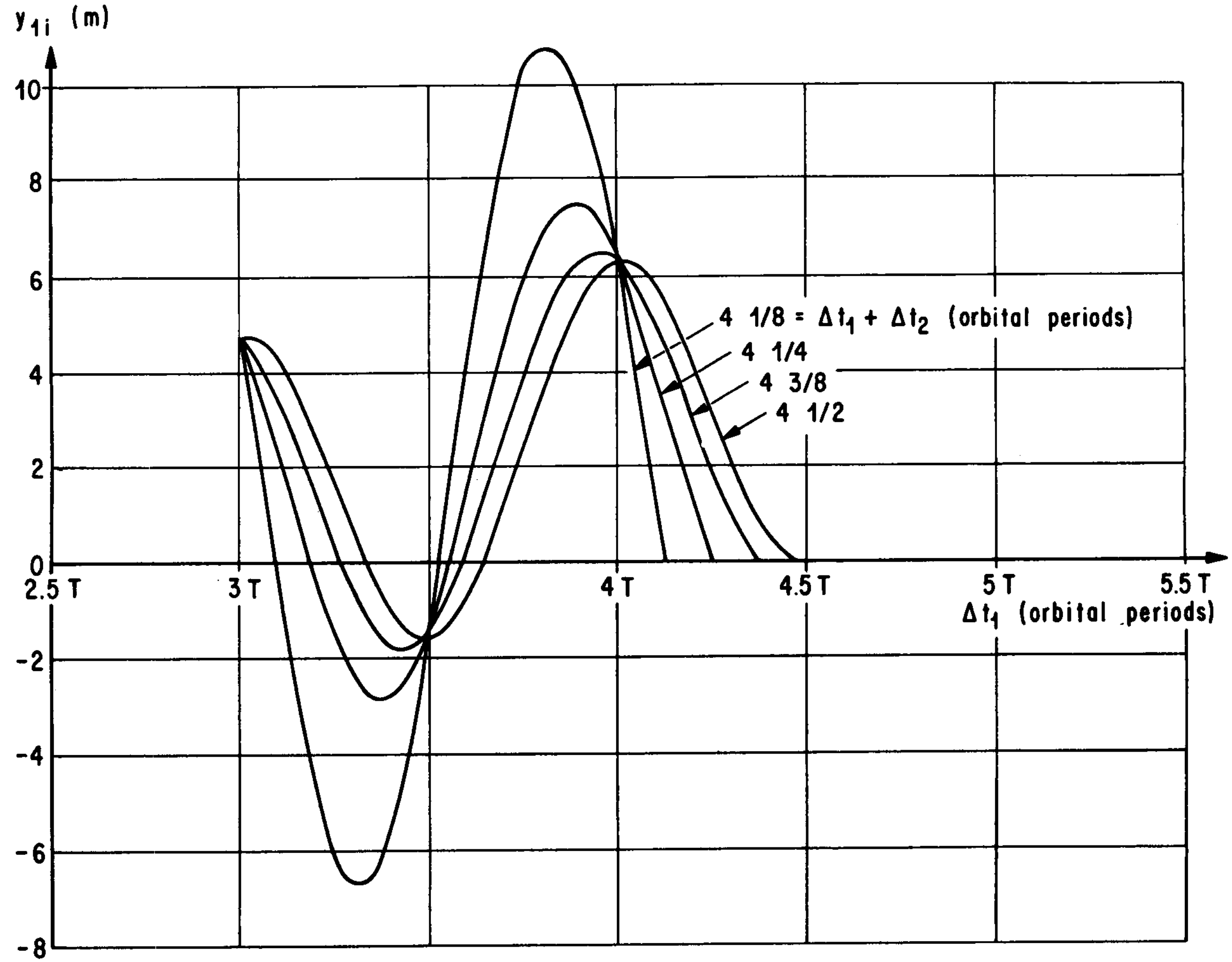

FIG. 13. CONTOURS OF EQUAL TOTAL CYCLE TIME ON THE PLANE (Δt_1, y_{1i})

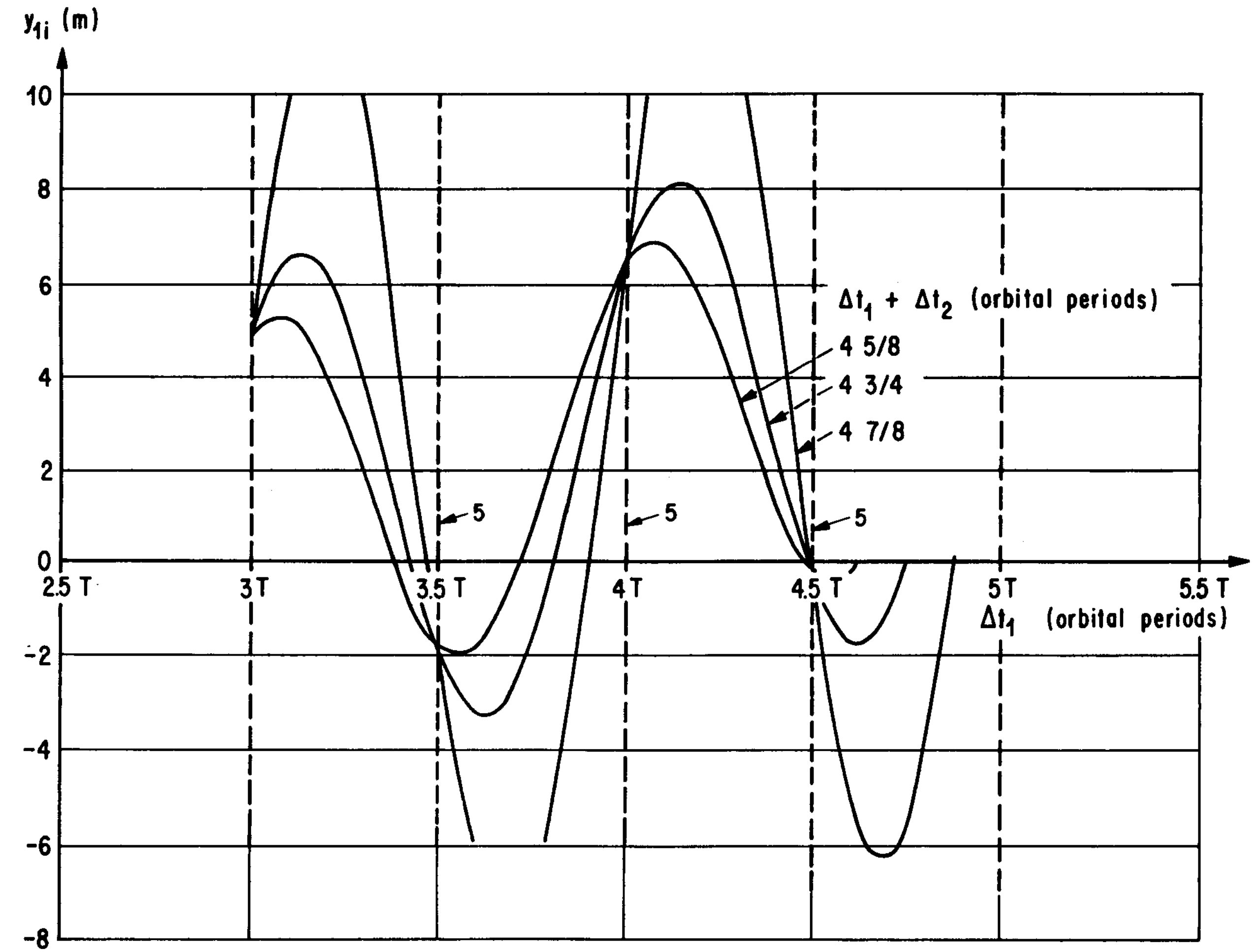

FIG. 14. CONTOURS OF EQUAL TOTAL CYCLE TIME ON THE PLANE $(\Delta t_1, y_{1i})$

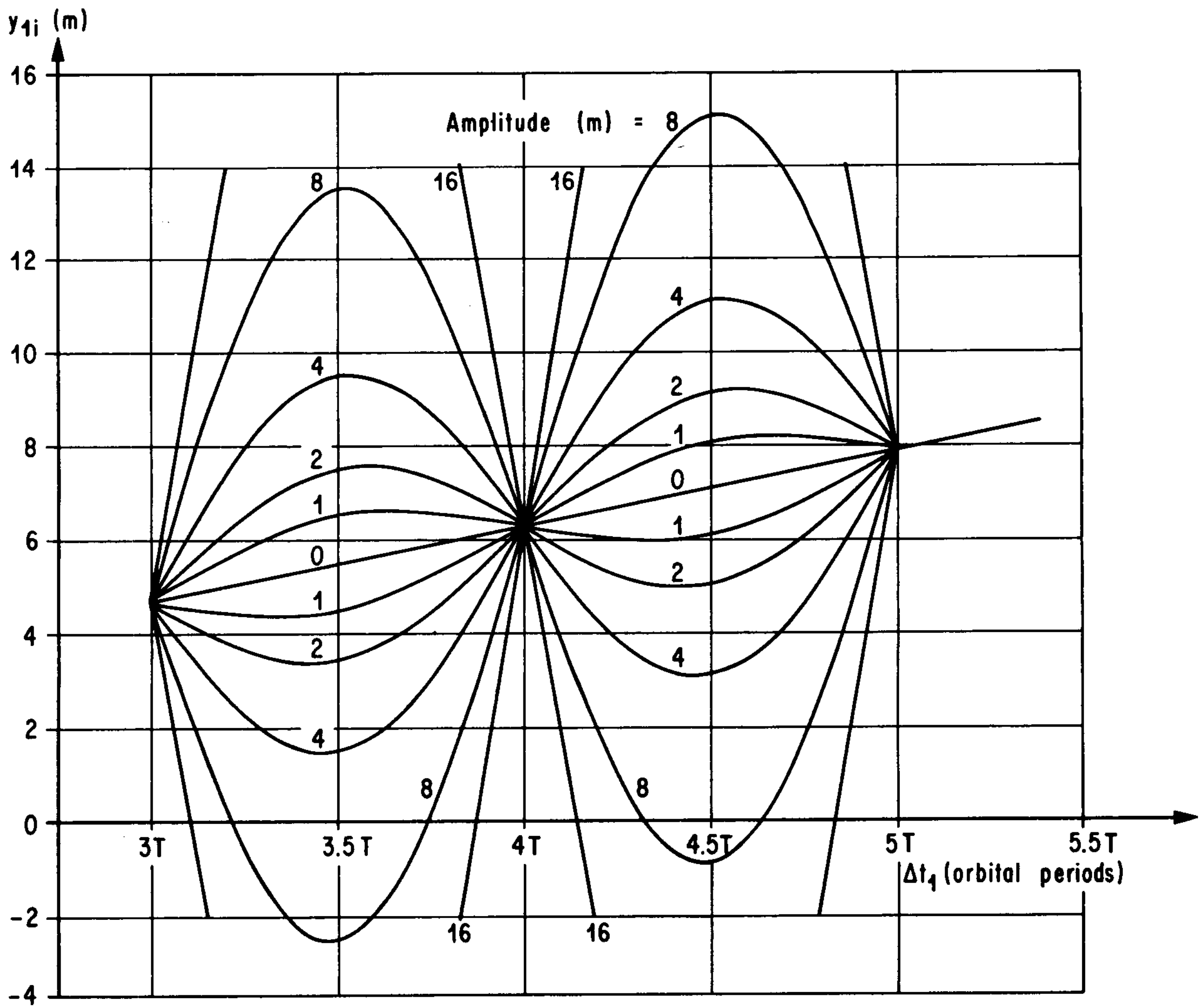

FIG. 15. CONTOURS OF EQUAL VERTICAL OSCILLATIONS ON THE PLANE (Δt_1, y_{1i})

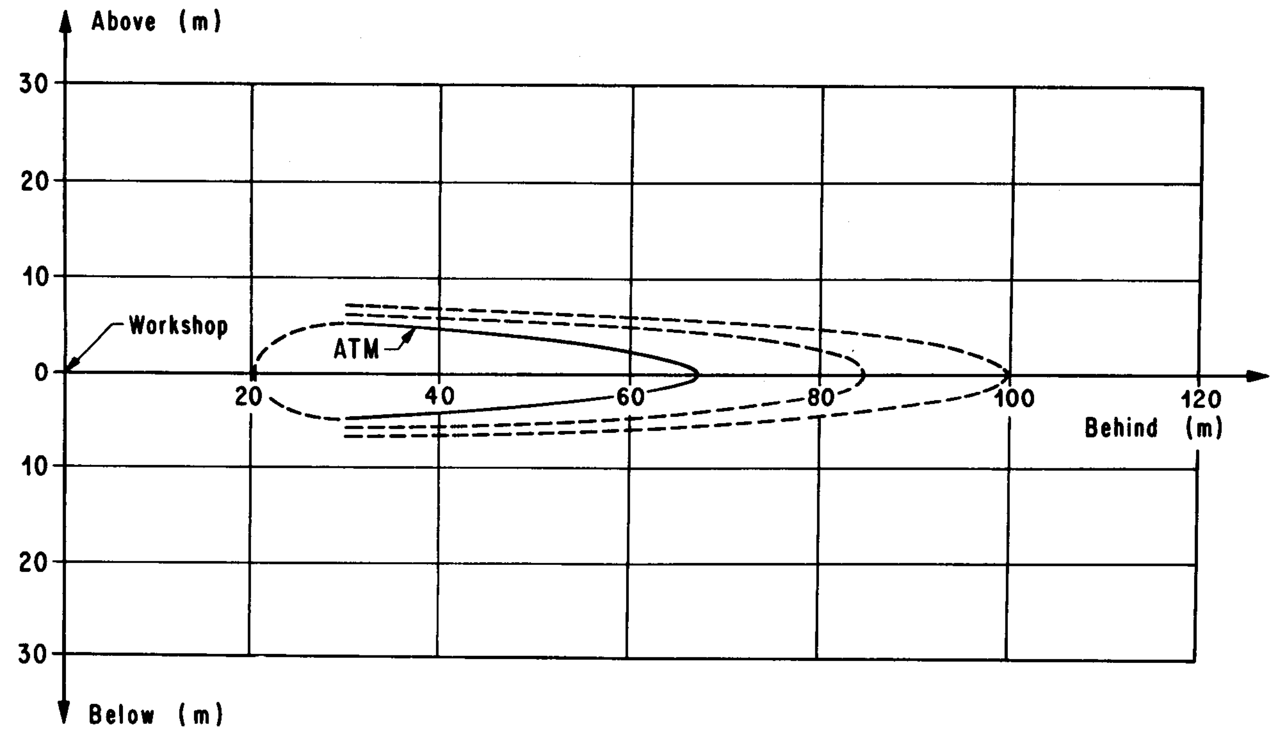

FIG. 16. ZERO OSCILLATION CASES

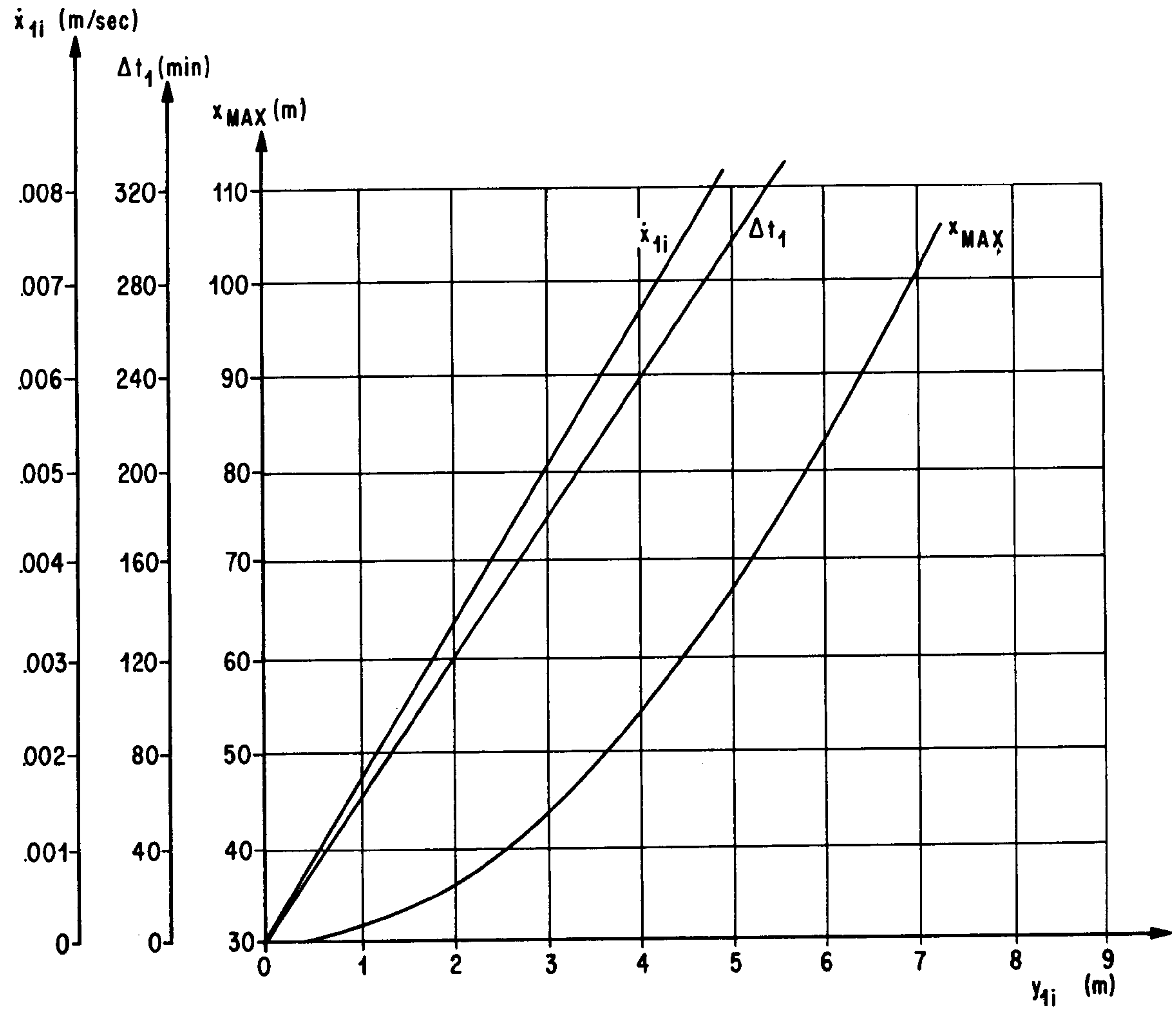

FIG. 17. ZERO OSCILLATION DATA

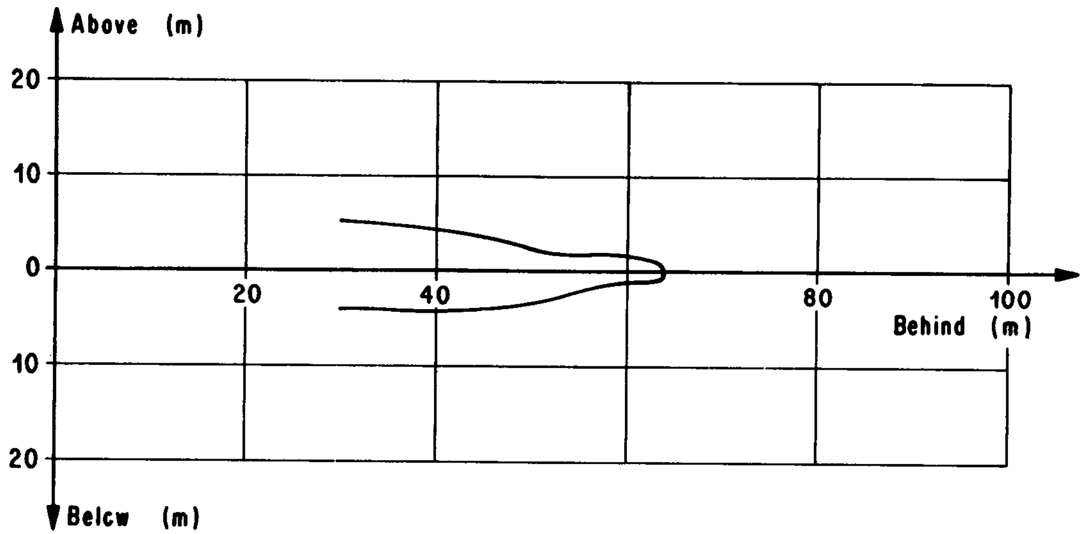

FIG. 18. RECIRCULARIZATION IMPULSE 10 % TOO SMALL

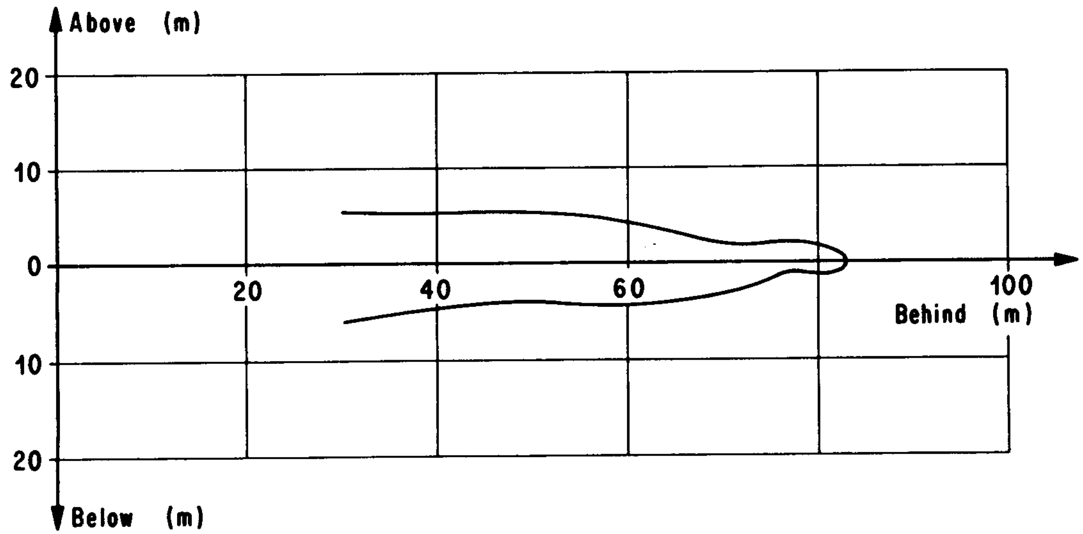

FIG. 19. RECIRCULARIZATION IMPULSE 10% TOO LARGE

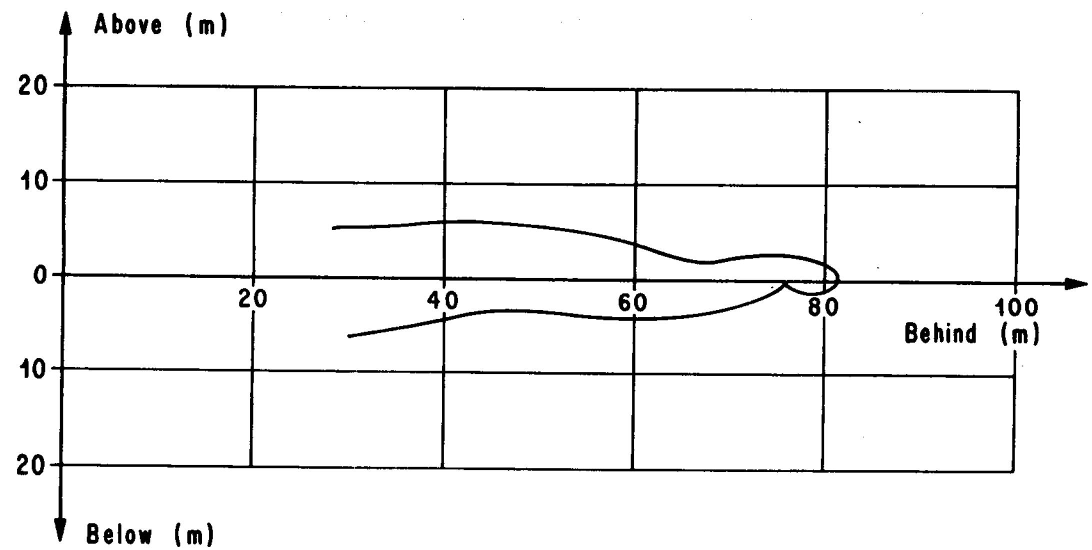

FIG. 20. RECIRCULARIZATION IMPULSE MADE 3 MINUTES EARLY

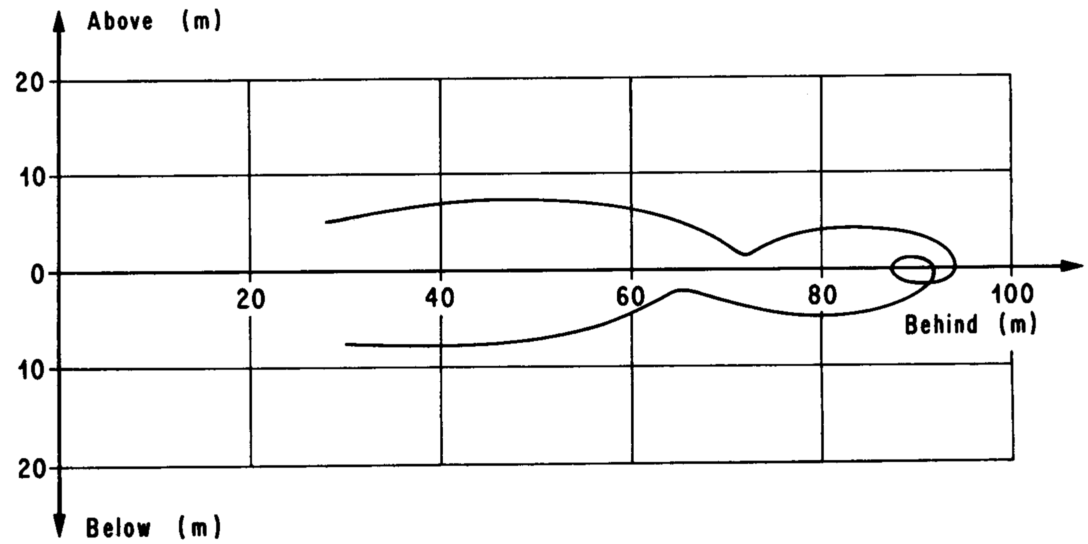

FIG. 21. RECIRCULARIZATION IMPULSE MADE 5 MINUTES EARLY

APPENDIX A

Solution to the Linearized Differential Equation

Equation (1) may be integrated once to yield

$$\dot{x} = Dt + 2\omega y - \frac{1}{3} c_o, \tag{A1}$$

where c_o is a constant of integration. Substituting equation (A1) into (2) and rearranging yields

$$\ddot{y} + \omega^2 y = \frac{2}{3} \omega c_o - 2\omega\, Dt. \tag{A2}$$

The solution to the homogeneous part is

$$y_h = -b_o \cos \omega t + a_o \sin \omega t,$$

where b_o and c_o are arbitrary constants. By inspection, the solution to the nonhomogeneous equation has the form

$$y = y_h + C_1 + C_2 t.$$

By differentiation and substitution into equation (A2), we obtain

$$\ddot{y}_h + \omega^2 y_h + \omega^2 C_1 + \omega^2 C_2 t = \frac{2}{3} \omega c_o - 2\omega\, Dt.$$

Therefore,

$$C_1 = \frac{2c_o}{3\omega},$$

$$C_2 = -\frac{2D}{\omega},$$

and

$$y = -b_o \cos \omega t + a_o \sin \omega t + \frac{2c_o}{3\omega} - \frac{2D}{\omega} t. \qquad \text{(A3)}$$

Substituting this into equation (A1) and integrating yields

$$x = -2b_o \sin \omega t - 2a_o \cos \omega t + c_o t - \frac{3}{2} Dt^2 + d_o, \qquad \text{(A4)}$$

where d_o is the constant of integration. By differentiating equations (A3) and (A4) with respect to time, $\dot{x}$ and $\dot{y}$ are obtained as functions of time:

$$\dot{x} = -2\omega b_o \cos \omega t + 2\omega a_o \sin \omega t + c_o - 3Dt \qquad \text{(A5)}$$

$$\dot{y} = \omega b_o \sin \omega t + \omega a_o \cos \omega t - \frac{2D}{\omega}. \qquad \text{(A6)}$$

The constants of integration may be expressed in terms of the conditions of position and velocity at time t = 0, x_o, y_o, $\dot{x}_o$, and $\dot{y}_o$:

$$a_o = \frac{1}{\omega}\left(\dot{y}_o + \frac{2D}{\omega}\right)$$

$$b_o = 3y_o - \frac{2\dot{x}_o}{\omega}$$

$$c_o = 6\omega y_o - 3\dot{x}_o$$

$$d_o = x_o + 2a_o.$$

APPENDIX B

The Conditions Necessary for a Repeatable Sequence

To discover what must be done to obtain a repeatable sequence, all of the equations which must be simultaneously satisfied must be written down. It turns out that the equations simplify somewhat if the time, $t = 0$, is placed half way into the initial trajectory; i.e., if Δt_1 is the time spent in the initial trajectory between pulls, then the trajectory begins at time, $t = -1/2\ \Delta t_1$, and ends at $t = 1/2\ \Delta t_1$. The initial conditions are as follows:

$$x_{1i} = x_o - \frac{1}{2} c_o \Delta t_1 - \frac{3}{8} D(\Delta t_1)^2 + 2b_o \sin \frac{1}{2} \omega\Delta t_1 + 2a_o\left(1 - \cos \frac{1}{2} \omega\Delta t_1\right) \tag{B1}$$

$$y_{1i} = \frac{2c_o}{3\omega} + \frac{D}{\omega} \Delta t_1 - b_o \cos \frac{1}{2} \omega\Delta t_1 - a_o \sin \frac{1}{2} \omega\Delta t_1 \tag{B2}$$

$$\dot{x}_{1i} = c_o + \frac{3}{2} D\Delta t_1 - 2\omega b_o \cos \frac{1}{2} \omega\Delta t_1 - 2\omega a_o \sin \frac{1}{2} \omega\Delta t_1 \tag{B3}$$

$$\dot{y}_{1i} = - \frac{2D}{\omega} - \omega b_o \sin \frac{1}{2} \omega\Delta t_1 + \omega a_o \cos \frac{1}{2} \omega\Delta t_1, \tag{B4}$$

where a_o, b_o, and c_o are defined as before.

The final conditions just before the impulse which initiate the transfer are

$$x_{1f} = x_o + \frac{1}{2} c_o \Delta t_1 - \frac{3}{8} D(\Delta t_1)^2 - 2b_o \sin \frac{1}{2} \omega\Delta t_1 + 2a_o\left(1 - \cos \frac{1}{2} \omega\Delta t_1\right) \tag{B5}$$

$$y_{1f} = \frac{2c_o}{3\omega} - \frac{D}{\omega} \Delta t_1 - b_o \cos \frac{1}{2} \omega\Delta t_1 + a_o \sin \frac{1}{2} \omega\Delta t_1 \tag{B6}$$

$$\dot{x}_{1f} = c_o - \frac{3}{2} D\Delta t_1 - 2\omega b_o \cos \frac{1}{2} \omega\Delta t_1 + 2\omega a_o \sin \frac{1}{2} \omega\Delta t_1 \tag{B7}$$

$$\dot{y}_{1f} = - \frac{2D}{\omega} + \omega b_o \sin \frac{1}{2} \omega\Delta t_1 + \omega a_o \cos \frac{1}{2} \omega\Delta t_1 \ . \tag{B8}$$

At this time, an impulse is made which changes the velocity by ΔV_1 in the direction of the tether, making the initial components of the velocity of the transfer conic

$$\dot{x}_{2i} = \dot{x}_{1f} - \Delta V_1 \frac{x_{1f}}{[x_{1f}^2 + y_{1f}^2]^{1/2}} \tag{B9}$$

$$\dot{y}_{2i} = \dot{y}_{1f} - \Delta V_1 \frac{y_{1f}}{[x_{1f}^2 + y_{1f}^2]^{1/2}} \ . \tag{B10}$$

For the transfer conic, if time is referenced to this point, after some time interval, Δt_2, it is desired that the satellite be back at the initial position of the first trajectory. Again, by using equations (A3) through (A6),

$$x_{1i} = x_{1f} + c'_o\Delta t_2 - \frac{3}{2} D(\Delta t_2)^2 - 2b'_o \sin \omega\Delta t_2 + 2a'_o(1 - \cos \omega\Delta t_2) \tag{B11}$$

$$y_{1i} = \frac{2c'_o}{3\omega} - \frac{2D}{\omega} \Delta t_2 - b'_o \cos \omega\Delta t_2 + a'_o \sin \omega\Delta t_2. \tag{B12}$$

The final velocities in the transfer conic will be

$$\dot{x}_{2f} = c'_o - 3D\Delta t_2 - 2\omega b'_o \cos \omega\Delta t_2 + 2\omega a'_o \sin \omega\Delta t_2 \tag{B13}$$

$$\dot{y}_{2f} = - \frac{2D}{\omega} + \omega b'_o \sin \omega\Delta t_2 + \omega a'_o \cos \omega\Delta t_2. \tag{B14}$$

In the last four equations,

$$a_o' = \frac{1}{\omega}\left(\dot{y}_{2i} + \frac{2D}{\omega}\right)$$

$$b_o' = 3y_{1f} - 2\,\frac{\dot{x}_{2i}}{\omega}$$

$$c_o' = 6\omega y_{1f} - 3\dot{x}_{2i}.$$

The initial velocity now must be obtained by making a second tug and changing the velocity by an amount ΔV_2.

$$\dot{x}_{1i} = \dot{x}_{2f} - \Delta V_2 \frac{x_{1i}}{[x_{1i}^2 + x_{1i}^2]^{1/2}} \qquad \text{(B15)}$$

$$\dot{y}_{1i} = \dot{y}_{2f} - \Delta V_2 \frac{y_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}}\,. \qquad \text{(B16)}$$

After substitution of the definitions for a_o, b_o, c_o, a_o', b_o', and c_o', there are 16 equations which must be satisfied by 20 variables. These are x_o, y_o, $\dot{x}_o$, $\dot{y}_o$, Δt_1, x_{1i}, y_{1i}, $\dot{x}_{1i}$, $\dot{y}_{1i}$, x_{1f}, y_{1f}, $\dot{x}_{1f}$, $\dot{y}_{1f}$, ΔV_1, $\dot{x}_{2i}$, $\dot{y}_{2i}$, Δt_2, $\dot{x}_{2f}$, $\dot{y}_{2f}$, and ΔV_2. Since there are four more variables than equations, four of the variables may be treated as parameters. Let these parameters be x_{1i}, y_{1i}, y_o, and $\dot{x}_o$. If, arbitrarily, y_o and $\dot{x}_o$ are set equal to zero, then $b_o = c_o = 0$. From equations (A3) through (A6), the equations of motion then become

$$x_1 = 2a_o\,(1 - \cos \omega t) - \frac{3}{2}\,Dt^2 + x_o \qquad \text{(B17)}$$

$$y_1 = a_o \sin\ t - \frac{2D}{\omega}\,t \qquad \text{(B18)}$$

$$\dot{x}_1 = 2\omega a_o \sin \omega t - 3Dt \qquad \text{(B19)}$$

$$\dot{y}_1 = \omega a_o \cos \omega t - \frac{2D}{\omega} . \qquad \text{(B20)}$$

As can be seen, this makes the trajectory symmetric about the horizontal axis; i.e., $x(t) = x(-t)$ and $y(t) = -y(-t)$. Also notice that $\dot{x}(t) = -\dot{x}(-t)$ and $\dot{y}(t) = \dot{y}(-t)$. Therefore, the initial and final conditions at the times $-1/2\ \Delta t_1$ and $1/2\ \Delta t_1$, respectively, are also symmetric: $x_{1f} = x_{1i}$, $y_{1f} = -y_{1i}$, $\dot{x}_{1f} = -\dot{x}_{1i}$, and $\dot{y}_{1f} = y_{1i}$. By solving equation (B1) for x_o, and equation (B2) for a_o, the variables x_o and $\dot{y}_o$ may be eliminated from the system of equations, if the resulting expressions are substituted into the other 16 equations. Noting that b_o and c_o are now zero, equations (B3) and (B4) become

$$\dot{x}_{1i} = 2\omega y_{1i} - \frac{1}{2} D\Delta t_1 \qquad \text{(B21)}$$

$$\dot{y}_{1i} = (-\omega y_{1i} + D\Delta t_1) \cot \frac{\omega\Delta t_1}{2} - 2\frac{D}{\omega} . \qquad \text{(B22)}$$

These two equations become the initial conditions which cause the motion to be symmetric. By using the symmetry conditions, x_{1f}, y_{1f}, $\dot{x}_{1f}$, and $\dot{y}_{1b}$, can be eliminated from equations (B9) through (B12):

$$\dot{x}_{2i} = -\dot{x}_{1i} - \Delta V_1 \frac{x_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}} \qquad \text{(B23)}$$

$$\dot{y}_{2i} = \dot{y}_{1i} + \Delta V_1 \frac{y_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}} \qquad \text{(B24)}$$

$$0 = c_o' \Delta t_2 - \frac{3}{2} D(\Delta t_2)^2 - 2b_o' \sin \omega\Delta t_2 + 2a_o'(1 - \cos \omega\Delta t_2) \qquad \text{(B25)}$$

$$y_{1i} = \frac{2c_o'}{\omega} - \frac{2D}{\omega} \Delta t_2 - b_o' \cos \omega\Delta t_2 + a_o' \sin \omega\Delta t_2, \qquad \text{(B26)}$$

where

$$a'_o = \frac{1}{\omega}\left(\dot{y}_{2i} + \frac{2D}{\omega}\right)$$

$$b'_o = -3y_{1i} - 2\,\frac{\dot{x}_{2i}}{\omega}$$

$$c'_o = -6\omega y_{1i} - 3\dot{x}_{2i}.$$

Using equations (B13) and (B14) to eliminate $\dot{x}_{2f}$ and $\dot{y}_{2f}$ from equations (B15) and (B16),

$$\dot{x}_{1i} = c'_o - 3D\Delta t_2 - 2\omega b'_o \cos \omega\Delta t_2 + 2\omega\, a'_o \sin \omega\Delta t_2 - \Delta V_2 \frac{x_{1i}}{[x^2_{1i} + y^2_{1i}]^{1/2}} \tag{B27}$$

$$\dot{y}_{1i} = -\frac{2D}{\omega} + \omega b'_o \sin \omega\Delta t_2 + \omega a'_o \cos \omega\Delta t_2 - \Delta V_2 \frac{y_{1i}}{[x^2_{1i} + y^2_{1i}]^{1/2}} . \tag{B28}$$

Now there are eight equations (B21) through (B27), with the two parameters, x_{1i} and y_{1i}, and eight variables, $\dot{x}_{1i}$, $\dot{y}_{1i}$, Δt_1, $\dot{x}_{2i}$, ΔV_1, $\dot{y}_{2i}$, Δt_2, and ΔV_2, having eliminated the others. Equations (B25) through (B28) insures that the initial position and velocity components are reproduced. Now, instead of making ΔV_1 of such a magnitude that the initial conditions are reproducible, consider the consequences of making it so that the new (transfer) trajectory is symmetric about the horizontal axis. Because of the symmetry after some time Δt_2, the body is back at the initial position of the first trajectory with velocity components symmetric to those of equations (B23) and (B24):

$$\dot{x}_{2f} = -\dot{x}_{2i} = \dot{x}_{1i} + \Delta V_1 \frac{x_{1i}}{[x^2_{1i} + y^2_{1i}]^{1/2}}$$

$$\dot{y}_{2f} = \dot{y}_{2i} = \dot{y}_{1i} + \Delta V_1 \frac{y_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}} \cdot$$

After the second pull on the tether, the velocity components are

$$\dot{x}_{new} = \dot{x}_{1i} + \Delta V_1 \frac{x_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}} - \Delta V_2 \frac{x_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}}$$

and

$$\dot{y}_{new} = \dot{y}_{1i} + \Delta V_1 \frac{y_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}} - \Delta V_2 \frac{y_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}} \quad .$$

Notice that if the second pull is made so that $\Delta V_2 = \Delta V_1$, then

$$\dot{x}_{new} = \dot{x}_{1i} \quad \text{and} \quad \dot{y}_{new} = \dot{y}_{1i};$$

i.e., the initial velocity has been reobtained. Therefore, the equations (B25) through (B28) insuring that the initial conditions are reobtained can be replaced by making ΔV_1 so that the trajectory is symmetric, waiting some time Δt_2, and making $\Delta V_2 = \Delta V_1$. To insure symmetry, ΔV_1 must be made so that $\dot{x}_{2i}$, and $\dot{y}_{2i}$ in equations (B23) and (B24) must satisfy equations similar to equations (B21) and (B22) after the time Δt_2:

$$\dot{x}_{2i} = 2\omega y_{1f} - \frac{1}{2} D\Delta t_2 = -2\omega y_{1i} - \frac{1}{2} D\Delta t_2 \tag{B29}$$

$$\dot{y}_{2i} = (-\omega y_{1f} + D\Delta t_2) \cot \frac{\omega\Delta t_2}{2} - \frac{2D}{\omega} = (\omega y_{1i} + D\Delta t_2) \cot \frac{\omega\Delta t_2}{2} - \frac{2D}{\omega}. \tag{B30}$$

Notice that the four equations (B25) through (B28) have been replaced by equations (B29) and (B30) and the condition $\Delta V_2 = \Delta V_1$. This indicates that there was some redundancy in the system of equations (B21) through (B28), and there is one more variable which may be arbitrarily treated as a parameter. Let it be Δt_1. Before parameterization, more variables and equations can be eliminated by replacing ΔV_2 by ΔV_1 and by eliminating $\dot{x}_{2i}$ and $\dot{y}_{2i}$ between equations (B23), (B24), (B29), and (B30). The system of equations then becomes

$$\dot{x}_{1i} = 2\omega y_{1i} - \frac{1}{2} D\Delta t_1$$

$$\dot{y}_{1i} = (-\omega y_{1i} + D\Delta t_1) \cot \frac{\omega\Delta t_1}{2} - 2 \frac{D}{\omega}$$

$$\dot{x}_{1i} + \Delta V_1 \frac{x_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}} = 2\omega y_{1i} + \frac{1}{2} D\Delta t_2$$

$$\dot{y}_{1i} + \Delta V_1 \frac{y_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}} = (\omega y_{1i} + D\Delta t_2) \cot \frac{\omega\Delta t_2}{2} - 2 \frac{D}{\omega} .$$

Eliminating $\dot{x}_{1i}$ and $\dot{y}_{1i}$ yields

$$\Delta V_1 \frac{x_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}} = \frac{1}{2} D(\Delta t_1 + \Delta t_2) \tag{B31}$$

$$\Delta V_1 \frac{y_{1i}}{[x_{1i}^2 + y_{1i}^2]^{1/2}} = (\omega y_{1i} + D\Delta t_2) \cot \frac{\omega\Delta t_2}{2} + (\omega y_{1i} - D\Delta t_1) \cot \frac{\omega\Delta t_1}{2} . \tag{B32}$$

Eliminating ΔV_1 yields

$$\frac{1}{2} D(\Delta t_1 + \Delta t_2) \frac{y_{1i}}{x_{1i}} = (\omega y_{1i} + D\Delta t_2) \cot \frac{\omega \Delta t_2}{2} + (\omega y_{1i} - D\Delta t_1) \cot \frac{\omega \Delta t_1}{2} . \tag{B33}$$

Thus, all of the pertinent variables can be expressed in terms of the parameters x_{1i}, y_{1i}, and Δt_1. Equations (B21) and (B22) give $\dot{x}_{1i}$ and $\dot{y}_{1i}$ in terms of y_{1i} and Δt_1. Equation (B33) can be numerically solved for Δt_2. After this is done, ΔV_1 can be found from equation (B31).

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A METHOD OF SOFT TETHER STATIONKEEPING

by B. S. Perrine, Jr.

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